

Improving the Lattice QCD Hamiltonian

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1. Introduction and Survey

Lepage's improvement scheme[1] is a major progress in the recent development of lattice QCD, opening the possibility to approach continuum physics on very coarse lattices. For the purpose of profiting from this scheme in the Hamiltonian formulation, the problem of improving the lattice Hamiltonian for the pure gauge theory is investigated. Following the procedure of Lepage, first a classically improved Kogut Susskind Hamiltonian is derived using a transfer matrix[2] or Legendre transformation[3] method. In this formulation, the color electric energy becomes an infinite series with long range terms. This deficiency can be cured by arguing with the non-uniqueness of the classically improved action yielding the result that the same order of improvement can be achieved by keeping only nearest neighbor interactions. A further tadpole improvement[1] of the Hamiltonian may be introduced straightforwardly. A final improvement in the sense of Lüscher and Weiss[4] involves a (Hamiltonian) perturbative calculation of suitable observables which will be deferred to the future. Here we will only discuss the qualitative structure of the occurring new Hamiltonian terms. An incorporation of the improved Hamiltonian into a coupled cluster calculation[5,6] of the spectrum is possible and should allow to obtain reliable results for the glueball spectrum in lower order than with the original Kogut Susskind Hamiltonian.

2. Hamiltonian Formalism

Starting from a lattice action written in terms of link variables $U_{x,\mu}$, the related Hamiltonian may be constructed by the following procedure. First one has to differentiate spacelike (a) and timelike (a_0) lattice spacings, and one has to introduce a temporal gauge: $U_{x,0} = 1$. The action may then be decomposed in the form (for details see [2])

$$S = \frac{a}{a_0} S^0 + \frac{a_0}{a} S^1 ,$$

where S^1 does not couple different times

$$S^1 = \sum_t S^1(U_t) .$$

Here, U_t stands for the set of (spacelike) link variables at fixed time t , $U_t = \{U_{\mathbf{x},t,j}\}$. For small a_0 , S^0 may be restricted to have the structure

$$S^0 = \sum_t S^0(U_t, V_t, V_t^\dagger) ,$$

where $V_t = \{U_{\mathbf{x},t+a_0,j} U_{\mathbf{x},t,j}^\dagger\}$. For $a_0 \rightarrow 0$ the elements of the set V_t are close to unity and may be characterized by the Lie algebra of $SU(N_c)$

$$U_{\mathbf{x},t+a_0,j} U_{\mathbf{x},t,j}^\dagger = 1 + i q_{\mathbf{x},j}(t) - \frac{1}{2} (q_{\mathbf{x},j}(t))^2$$

where $q_{\mathbf{x},j}(t) = \sum_b q_{\mathbf{x},j,b}(t) T^b$. The general structure of S^0 is then

$$S^0 = \sum_t \sum_{\alpha,\beta} \frac{1}{2} q_\alpha(t) M_{\alpha,\beta}(U_t) q_\beta(t) ,$$

$$\alpha = (\mathbf{x}, j, b) . \quad (1)$$

The *Hamiltonian* related to the action S is then defined on Hilbertspace functions $\psi(U)$ (t fixed) and is given by

$$\begin{aligned} H &= T + V \\ T &= \frac{1}{2a} \sum_{\alpha, \beta} E_{\alpha} M_{\alpha, \beta}^{-1}(U) E_{\beta} \\ V &= \frac{1}{a} S^1(U), \end{aligned} \quad (2)$$

where $E_{\mathbf{x}, j} = E_{\mathbf{x}, j, b} T^b$ is conjugate to $U_{\mathbf{x}, j}$. The validity of these formulae can be derived using the transfer matrix[2] or the Legendre transformation[3] method.

The simplest example for the action S is the Wilson action $S_{Wilson} = \beta \sum_{x, \mu > \nu} (1 - P_{\mu\nu}(x))$ ($P_{\mu\nu}$ is the standard plaquette parallel transporter.) yielding $M_{\alpha, \beta} = g^{-2} \delta_{\alpha, \beta}$ and the Kogut-Susskind Hamiltonian

$$H_{KS} = \frac{g^2}{2a} \sum_{\alpha} E_{\alpha} E_{\alpha} + \frac{2N_c}{ag^2} \sum_{\mathbf{x}, j > k} (1 - P_{jk}(\mathbf{x}))$$

3. The Classically Improved Hamiltonian

The classical $O(a^2)$ errors of the Wilson action are compensated by introducing the improved action[1] (we disregard constant terms)

$$S_{cl} = -\beta \sum_{x, \mu > \nu} \left[\frac{5}{3} P_{\mu\nu}(x) - \frac{1}{12} (R_{\mu\nu}(x) + R_{\nu\mu}(x)) \right]$$

where $R_{\mu\nu}$ is the $2a \times a$ "rectangle loop parallel transporter".

Differentiating space- and time-like lattice spacings the improved potential part of the Hamiltonian is given by the space part of S_{cl}

$$V_{cl} = -\frac{\beta}{a} \sum_{\mathbf{x}, j > k} \left(\frac{5}{3} P_{jk}(\mathbf{x}) - \frac{1}{12} (R_{jk}(\mathbf{x}) + R_{kj}(\mathbf{x})) \right)$$

The kinetic part involves a non-trivial computation: The S^0 -part of S_{cl} contains the time-like terms P_{i0} and R_{i0} which have up to errors of the order $O(a^4, a_0^2)$ the continuum limit behaviour (in the limit $a_0 \rightarrow 0$ errors of $O(a_0^4)$ can be disregarded)

$$P_{i0} = -\frac{a_0^2 a^2}{\beta} \left(Tr F_{i0}^2 + \frac{a^2}{12} Tr F_{i0} (D_j^2) F_{i0} \right)$$

$$R_{i0} = -\frac{a_0^2 a^2}{\beta} \left(4 Tr F_{i0}^2 + \frac{16a^2}{12} Tr F_{i0} (D_j^2) F_{i0} \right)$$

Consequently, a classical improvement of Wilson's S^0 is given by

$$S_{cl}^0 = -\beta \frac{4}{3} \left(\sum_{x, j} P_{0j}(x) - \frac{1}{16} \sum_{x, j} R_{0j}(x) \right)$$

With temporal gauge fixing and expanding according to (1) this yields for the matrix M

$$\begin{aligned} M_{\alpha, \alpha'} &= \frac{4g^2}{5} [\delta_{\alpha, \alpha'} \\ &\quad - \frac{1}{20} \delta(\mathbf{x}', \mathbf{x} + j) \delta_{j, j'} Tr U_{\mathbf{x}, j}^{\dagger} T^a U_{\mathbf{x}, j} T^{a'}] \end{aligned}$$

The inversion of the matrix M yields for the kinetic part T of the classically improved Hamiltonian an infinite number of terms

$$\begin{aligned} M_{\alpha, \alpha'}^{-1} &= \frac{5}{4g^2} [\delta_{\alpha, \alpha'} + \\ &\quad \sum_{n=1}^{\infty} \left(\frac{1}{20} \right)^n \delta(\mathbf{x}', \mathbf{x} + nj) \delta_{j, j'} Tr U_{\mathbf{x}, nj}^{\dagger} T^a U_{\mathbf{x}, nj} T^{a'}] \end{aligned}$$

where $U_{\mathbf{x}, nj}$ is the straight line parallel transporter from \mathbf{x} to $\mathbf{x} + nj$,

$U_{\mathbf{x}, nj} = U_{\mathbf{x}, j} U_{\mathbf{x}+j, j} \dots U_{\mathbf{x}+(n-1)j, j}$. Note that the combination of the parallel transporters $U_{\mathbf{x}, nj}$ and the conjugate operators E_{α} in (2) guarantees the gauge invariance of T .

4. Classically Improved Hamiltonian with Finite Number of Terms

The occurrence of infinite number of terms in T can be avoided by invoking the fact that the classically improved action given by demanding cancellation of errors up to the order $O(a^4)$ is not uniquely fixed. Herefore we note that the term

$$\begin{aligned} S_{nj} &= \sum_{\mathbf{x}, a} q_{\mathbf{x}, j, a} \delta(\mathbf{x}, \mathbf{x} + nj) \delta_{j, j'} \\ &\quad \times Tr U_{\mathbf{x}, nj}^{\dagger} T^a U_{\mathbf{x}, nj} T^{a'} q_{\mathbf{x}+nj, j', a'} \end{aligned} \quad (3)$$

emerges from a generalization of the $2a \times a_0$ loop term R_{j0} to a $na \times a_0$ loop parallel transporter $R_{nj,0}$ of the type

$$R_{nj,0} = \frac{1}{N_c} Re Tr \quad \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}$$

in the sense that we have with temporal gauge fixing and for small a_0

$$1 - R_{nj,0} = \frac{a}{\beta} \left(\sum_{\mathbf{x},a} q_{\mathbf{x},j,a} q_{\mathbf{x},j,a} + S_{nj} \right)$$

The continuum limit behaviour is in this case

$$R_{nj,0} = -\frac{a_0^2 a^2}{\beta} [4Tr F_{0j}^2 + \frac{(3n^2 + 1)a^2}{3} Tr F_{0j}(D_j^2) F_{0j} + O(a^4, a_0^2)]$$

Consequently, the ansatz

$$S^0 = -\frac{\beta}{A} \left(B \sum_{x,j} P_{0j}(x) + \sum_{n=1}^{\infty} C^n \sum_{x,j} R_{nj,0}(x) \right)$$

is legitimate for a consistent continuum limit, and yields with $B = 1 - \sum_{n=1}^{\infty} C^n$ for M a geometrical series

$$M_{\alpha,\alpha'} = g^{-2} A^{-1} [\delta_{\alpha,\alpha'} + \sum_n C^n \delta(\mathbf{x}', \mathbf{x} + nj) \delta_{j,j'} Tr U_{\mathbf{x},nj}^\dagger T^a U_{\mathbf{x},nj} T^{a'}]$$

Hence M^{-1} has only two terms

$$M_{\mathbf{x}ja;\mathbf{x}'j'a'}^{-1} = g^2 A [\delta_{\mathbf{x}ja;\mathbf{x}'j'a'} - C \delta(\mathbf{x}', \mathbf{x} + j) \delta_{j,j'} Tr U_{\mathbf{x},j}^\dagger T^a U_{\mathbf{x},j} T^{a'}] \quad (4)$$

A classical improvement of T to the order $O(a^4)$ is then obtained for $A = (1 + c^2)/(1 - c^2)$ and $C = 2c/(1 + c^2)$ where $c \simeq -.101$ is a solution of $\sum_{n=0}^{\infty} (1 + 6n^2)c^n = .5$.

5. Further Improvements

The *tadpole improvement* may be defined as in the Lagrangian case[1] by replacing $U_{\mathbf{x},j}$ by $U_{\mathbf{x},j}/u_0$ where the quantity u_0 may be computed from the vacuum expectation value of the plaquette operator via $u_0 = \langle 0 | P_{jk} | 0 \rangle^{1/4}$. For a Lüscher Weisz improvement[4], a new perturbative, Hamiltonian calculation of suitable observables has to be performed which has still to be done. We expect that the type of additional terms occurring in the Hamiltonian formulation correspond to those of Lepage. The potential part of

the improved Hamiltonian would then get corrections like the action of Lepage (see eq. (73) of ref.[1]). It might be interesting to note, however, that the time-like “parallelogram plaquettes” (eq. (72) of ref.[1]) yield contributions to T which are quadratic and linear in the operators E_α combined with non-straight parallel-transporters (plaquettes in lowest order). Details will be presented elsewhere[7].

XQL and SHG were supported by the National Natural Science Foundation and National Education Committee. H.K. was supported by NSERC Canada.

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